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Soft emissions and the equivalence of BFKL and CCFM final states

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Abstract: This article demonstrates that the BFKL and CCFM equations, despite their different physical content, lead to equivalent results for any final-state observable at leading single-logarithmic order. A novel and fundamental element is the treatment also of the soft ($z \to 1$ divergent) part of the splitting function in the CCFM equation.

Keywords: QCD, Deep Inelastic Scattering, Jets.

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Contents

1. Introduction

For the study of the properties of small-x deep inelastic scattering collisions, it is desirable to sum leading logarithmic (LL) terms, $(\alpha_s \ln x)^n$ of the perturbation series. This is generally done using the BFKL equation [1]. Formally its validity is guaranteed only for inclusive quantities, because its derivation relies on the dominance of multi-Regge kinematics.

To guarantee leading logarithms of x also for exclusive quantities, it is necessary to take into account the QCD coherence and soft radiation (that from the $z \to 1$ divergent part of the splitting function). This is done in the CCFM equation [2, 3]. For inclusive quantities it can be demonstrated to give the same results as the BFKL equation to LL level. For exclusive quantities this is not necessarily the case.

A few years ago Marchesini [4] showed that multi-jet rates do indeed differ in the BFKL and CCFM approaches: the latter involving $(\alpha_s \ln^2 x)^n$ factors, the former not. More recently Forshaw and Sabio Vera [5] considered multi-jet rates with the extra condition that the jets should be resolved (i.e. that their transverse momentum should be larger than a certain resolution parameter μ_R), and demonstrated to order α_s^3 that the BFKL and CCFM jet-rates were identical. This result was then extended to all orders by Webber [6].

The results [5,6] were derived in the double-logarithmic (DL) approximation, i.e. considering only powers of α_s that are accompanied by two logarithms, at least one of which is a $\ln x$. Additionally, in all cases, the CCFM equation was used without the $z \to 1$ divergent (soft) part of the splitting function.

A number of questions arise. Firstly, taking the limit of $\mu_R \to 0$ one should go from the result of Forshaw, Sabio Vera and Webber to that of Marchesini — the basic logarithmic structure being different, it is not clear how this comes about. The resolution of this puzzle comes through the consideration of formally subleading terms $\alpha_s \ln^2 Q/\mu_R$, which must be resummed when taking the limit $\mu_R \to 0$, and which give a continuous transition from the case of BFKL and CCFM results being equivalent, to that of their being different. This is presented in section 2.

A second question is the importance of the $z \to 1$ divergent part of the splitting function in the CCFM equation. In previous studies this has not been examined. In section 3 it will be shown that its inclusion leads to *all* BFKL and CCFM final-state properties being identical in the DL approximation.[1] It is also demonstrated that no potentially dangerous $\alpha_s \ln^2 \mu_R$ terms arise anymore (either in CCFM or BFKL), except some associated with the end of the branching chain.

Finally in section 4 the above result is extended to single logarithmic accuracy (i.e. terms where all powers of α_s are accompanied by a $\ln x$ factor, but not necessarily by any other logarithm). For an actual calculation of final-state properties to single-logarithmic accuracy, the reader is referred to [8].

2. Dividing up the double-logarithmic physics

2.1 BFKL

Let us start with the BFKL equation. We will consider ladders with kinematics labelled as in figure 1. The exchanged gluon i has longitudinal momentum fraction x_i, and transverse momentum k_i. We define also $z_i = x_i/x_{i-1}$. The transverse momentum coming into the chain is taken to be zero. Emitted gluons have transverse momentum $q_{t,i}$ and longitudinal momentum fraction $x_{i-1}(1 - z_i)$.

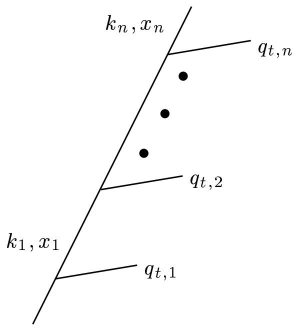

Figure 1: kinematics.

The unintegrated gluon density is given by

$$\mathcal{F}(x,k) = \sum_{n=0}^{\infty} \prod_{i=1}^{n} \left(\bar{\alpha}_s \int \frac{dz_i}{z_i} \int_{\mu} \frac{d^2\vec{q}_{t,i}}{\pi q_{t,i}^2} \Delta(z_i, k_i) \right) \times \\ \times \delta\!\left(x - \prod_{i=1}^{n} z_i \right) \delta^2\!\left(\vec{k} + \sum_{i=1}^{n} \vec{q}_{t,i} \right), \quad (2.1)$$

where n is the number of emissions, μ is a collinear cutoff and $\bar{\alpha}_s = \alpha_s C_A/\pi$. For

[1] This addresses also the issue, raised in [6,7], of whether differences might arise between BFKL and CCFM for *correlations* between resolved jets.

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$n = 0$, the product of z_i's should be interpreted as being equal to 1, so that the initial condition is $\delta(x-1)\delta^2(\vec{k})$. The virtual corrections are contained in the form factor $\Delta(z,k)$:

$$\ln \Delta(z,k) = -2\bar{\alpha}_{\mathrm{s}} \ln \frac{1}{z}\, \ln \frac{k}{\mu}\,, \tag{2.2}$$

and the k_i are given by

$$\vec{k}_i = -\sum_{j=1}^{i} \vec{q}_{t,i}\,, \qquad k = k_n\,. \tag{2.3}$$

Since we have two large quantities, $\ln k/\mu$ and $\ln 1/x$, it is of interest to carry out a double logarithmic expansion, i.e. concentrating on terms where each power of α_{s} is accompanied by two logarithms (in the BFKL case always the product $\ln 1/x\, \ln k/\mu$).

Each branching is guaranteed to give a factor $\alpha_{\mathrm{s}} \ln 1/x$. To obtain an extra (transverse) logarithm, the branching has to be sensitive to a ratio of transverse scales. The two such kinds of branching are those that increase the exchanged momentum k (k-changing emissions), $k_i \simeq q_{t,i} \gg k_{i-1}$, which give double logarithms in the total cross section; and those that don't change it at all (k-conserving emissions), $k_i \simeq k_{i-1} \gg q_{t,i}$, giving double logarithms only in the final state.

All remaining kinds of branching give just a factor of $\alpha_{\mathrm{s}} \ln 1/x$, and so can be neglected to double-logarithmic accuracy. Thus (2.1) becomes

$$\begin{aligned}\mathcal{F}(x,k) = \sum_{n=0}^{\infty}\prod_{i=1}^{n}\left(\bar{\alpha}_{\mathrm{s}}\int\frac{dz_i}{z_i}\int_{\mu}\frac{d^2\vec{q}_{t,i}}{\pi q_{t,i}^2}\Delta(z_i,k_i)\right)\times \\ \times\,\delta\!\left(x-\prod_{i=1}^{n}z_i\right)\delta^2\!\Big(\vec{k}+\max(\vec{q}_{t,1},\dots,\vec{q}_{t,n})\Big)\,,\end{aligned} \tag{2.4}$$

where the k_i are now defined as

$$k_i = -\max(\vec{q}_{t,1},\dots,\vec{q}_{t,i})\,. \tag{2.5}$$

We then note that

$$1 = \sum_{m=0}^{\infty}\prod_{\ell=1}^{m}\left(\bar{\alpha}_{\mathrm{s}}\int\frac{d\zeta_\ell}{\zeta_\ell}\int_{\mu}^{k}\frac{d^2\vec{\rho}_{t,\ell}}{\pi\rho_{t,\ell}^2}\Delta(\zeta_\ell,k)\right)\Delta(z/\zeta_{\Pi},k)\,\Theta(\zeta_{\Pi}-z)\,, \tag{2.6}$$

where z and k can take any value and

$$m = 0: \qquad \zeta_{\Pi} = 1\,, \tag{2.7a}$$

$$m \ge 1: \qquad \zeta_{\Pi} = \prod_{\ell=1}^{m}\zeta_\ell\,. \tag{2.7b}$$

The relation (2.6) is easily verified. Its significance is that k-conserving emissions are 'probability conserving': they are exactly compensated for by the virtual corrections.

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Let us now rewrite (2.4) with k-changing emissions and k-conserving emissions in separate sums,

$$\mathcal{F}(x,k) = \sum_{n=0}^{\infty} \delta^2\Big(\vec{k}+\vec{q}_{t,n}\Big) \prod_{i=1}^{n} \Bigg[\left(\bar{\alpha}_{\mathrm{s}} \int \frac{dz_i}{z_i} \int_{k_{i-1}} \frac{d^2\vec{q}_{t,i}}{\pi q_{t,i}^2} \right) \delta\Bigg(x - \prod_{i=1}^{n} z_i\Bigg) \times \\ \times \sum_{m=0}^{\infty} \prod_{\ell=1}^{m} \left(\bar{\alpha}_{\mathrm{s}} \int \frac{d\zeta_\ell}{\zeta_\ell} \int_{\mu}^{k_i} \frac{d^2\vec{\rho}_{t,\ell}}{\pi \rho_{t,\ell}^2} \Delta(\zeta_\ell, k_i) \right) \Delta(z_i/\zeta_{\Pi_i}, k_i)\, \Theta(\zeta_{\Pi_i} - z_i) \Bigg], \tag{2.8}$$

where ζ_{Π_i} is defined in analogy with (2.7), $k_0 = \mu$ and $k_i = q_{t,i}$. Exploiting (2.6) to eliminate the second line, one sees that the double-logarithmic BFKL cross section is given by

$$\mathcal{F}(x,k) = \sum_{n=0}^{\infty} \prod_{i=1}^{n} \left(\bar{\alpha}_{\mathrm{s}} \int \frac{dz_i}{z_i} \int_{k_{i-1}} \frac{d^2\vec{q}_{t,i}}{\pi q_{t,i}^2} \right) \delta\Bigg(x - \prod_{i=1}^{n} z_i\Bigg) \delta^2\Big(\vec{k}+\vec{q}_{t,n}\Big), \tag{2.9}$$

which is just the double-logarithmic DGLAP (ordered chain) [9] result for the gluon density. To obtain the full BFKL final-state prediction one then 'dresses' the ordered chain (figure 2) by putting back in a set of k-conserving emissions after each k-changing emission, i.e. by replacing the second line of (2.8).

The rapidity of each emission i is given by

$$\eta_i = \ln \frac{q_{t,i}}{x_{i-1}(1-z_i)}. \tag{2.10}$$

The procedure of removing (or adding back in) the k-conserving emissions modifies the z_i's for the k-changing emissions, but not the x_{i-1} values. Since z_i is anyway much less than 1, $1 - z_i$ changes by a negligible (next-to-leading) amount — so strong ordering ensures that rapidities are essentially unaffected by the removal and insertion of a subset of emissions, allowing one to safely use this technique for examining properties of the final state.

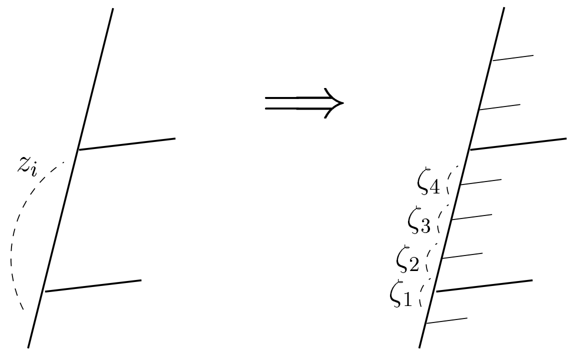

Figure 2: Adding low-q_t emissions to a strongly ordered chain: $\zeta_1 \cdots \zeta_4 = z_i$.

2.2 CCFM

The CCFM gluon density is given by

$$\mathcal{A}(x,k,p) = \sum_{n=0}^{\infty} \prod_{i=1}^{n} \left(\bar{\alpha}_{\mathrm{s}} \int \frac{dz_i}{z_i} \int \frac{d^2\vec{q}_i}{\pi q_i^2} \Delta(z_i, k_i, q_i)\, \Theta(q_i - z_{i-1}q_{i-1})\, \Theta(q_{t,i} - \mu) \right) \times \\ \times \Theta(p - z_n q_n)\, \delta\Bigg(x - \prod_{i=1}^{n} z_i\Bigg) \delta^2\Bigg(\vec{k} + \sum_{i=1}^{n} \vec{q}_{t,i}\Bigg). \tag{2.11}$$

Angular ordering is embodied by the factors $\Theta(q_i - z_{i-1}q_{i-1})$ (absent for $i = 1$). A maximum angle is introduced through the dependence on the third variable, p. The k_i are defined as in (2.3).

Relative to the BFKL equation, the virtual corrections differ so as to take into account the angular ordering, and are given by the non-Sudakov form factor $\Delta(z, k, q)$:

$$\ln\Delta(z,k,q) = -2\bar{\alpha}_s \int_z^1 \frac{d\zeta}{\zeta}\int\frac{d\rho}{\rho}\,\Theta(k-\rho)\,\Theta(\rho-\zeta q)\,\Theta(\rho-\mu) \tag{2.12a}$$

$$= -\bar{\alpha}_s\left(\ln^2\frac{1}{z} + 2\ln\frac{1}{z}\ln\frac{k}{q}\right), \qquad q<k, \quad zq>\mu\,. \tag{2.12b}$$

It is to be noted that q is a rescaled transverse momentum, $q = q_t/(1-z)$. Since we have only the $1/z$ part of the splitting function, we work in the limit of $z \ll 1$ and so the difference between q and q_t can be neglected.[2]

As in the BFKL case, when considering the DL limit, we can replace $\sum \vec{q}_{t,i}$ in the δ-function by the largest of the $q_{t,i}$:

$$\mathcal{A}(x,k,p) = \sum_{n=0}^{\infty}\prod_{i=1}^{n}\left(\bar{\alpha}_s\int\frac{dz_i}{z_i}\int\frac{d^2\vec{q}_i}{\pi q_i^2}\,\Delta(z_i,k_i,q_i)\,\Theta(q_i - z_{i-1}q_{i-1})\,\Theta(q_{t,i}-\mu)\right)\times$$
$$\times\,\Theta(p - z_n q_n)\,\delta\left(x - \prod_{i=1}^{n} z_i\right)\delta^2\Big(\vec{k} + \max(\vec{q}_{t,1},\ldots,\vec{q}_{t,n})\Big)\,, \tag{2.13}$$

with the k_i now defined as in (2.5). The next step is to note that

$$1 = \sum_{m=0}^{\infty}\prod_{\ell=1}^{m}\left(\bar{\alpha}_s\int\frac{d\zeta_\ell}{\zeta_\ell}\int^k\frac{d^2\vec{\rho}_\ell}{\pi\rho_\ell^2}\,\Delta(\zeta_\ell,k,\rho_\ell)\,\Theta(\rho_\ell - \zeta_{\ell-1}\rho_{\ell-1})\,\Theta(\rho_\ell-\mu)\right)\times$$
$$\times\,\Delta(z/\zeta_\Pi,k,\rho_0)\,\Theta(\zeta_\Pi - z)\,, \tag{2.14}$$

where ζ_Π is defined as in the BFKL case (2.7), $\zeta_0 = z/\zeta_\Pi$, and z, k and $\rho_0 \le k$ can take any values; ρ is a rescaled momentum, and $\rho_t = (1-\zeta)\rho$ is the corresponding transverse momentum (at our accuracy, they are not distinguishable for hard emissions, hence we are allowed to write $\Theta(\rho_\ell - \mu)$ rather than the slightly more correct $\Theta(\rho_{t,\ell} - \mu)$). To demonstrate the relation (2.14) term by term is quite difficult. Instead one can see that it holds because it relates to a probabilistic branching — and the total probability of all possible states is 1.

As in the BFKL case, the aim is now to show that the cross section is determined only by the k-changing emissions. To be able to perform the sum (2.14) after each k-changing emission, there are three conditions. Let us label a pair of successive k-changing emissions as a and b. The first condition is $\rho_0 \le k_a$, and it is satisfied because $\rho_0 \equiv q_a \simeq k_a$. The second condition is that the angular ordering of the

[2] As can be any factors of $1-z$. However when we introduce soft emissions, later on, it will be vital to retain their associated $1-z$ factors.

next k-changing emission should not cut out any piece of the sum (2.14): this is guaranteed since $q_b > k_a$, so that the angular ordering condition $z_a/\zeta_\Pi \rho_m < q_b$ is by definition satisfied (recall that $\rho_m < k_a$). Finally, we want to carry out the complete sum (2.14) after the last k-changing emission — for this to be possible we require that $p \geq k$.

Therefore one can extract the k-conserving emissions from the calculation of the cross section and obtain

$$\mathcal{A}(x, k, p \geq k) = \sum_{n=0}^{\infty} \prod_{i=1}^{n} \left(\bar{\alpha}_s \int \frac{dz_i}{z_i} \int_{\max(k_{i-1},\mu)} \frac{d^2\vec{q}_{t,i}}{\pi q_{t,i}^2} \right) \times \\ \times \Theta(p - z_n q_n)\, \delta\!\left(x - \prod_{i=1}^{n} z_i \right) \delta^2\!\left(\vec{k} + \vec{q}_{t,n}\right). \tag{2.15}$$

This is the same result as in the BFKL case (2.9).

To obtain the correct final state one must reintroduce after each k-changing emission a set of k-conserving emissions as given by (2.14).

2.3 BFKL and CCFM final states

Since the k-changing parts of the BFKL and CCFM equations are identical, it suffices, at least for now, to consider the k-conserving parts, (2.6) and (2.14) respectively.

The BFKL case is extremely straightforward. The emission of a gluon is a Poissonian type process, so that emissions are independent and

$$\left\langle \frac{dn}{d \ln q_t \, d \ln 1/x} \right\rangle = 2\bar{\alpha}_s\,, \tag{2.16}$$

where x is the longitudinal momentum fraction of the emission.

The CCFM case is more complex. The phase-space for the first emission differs from that in the BFKL case by a triangular region. The area of the difference is

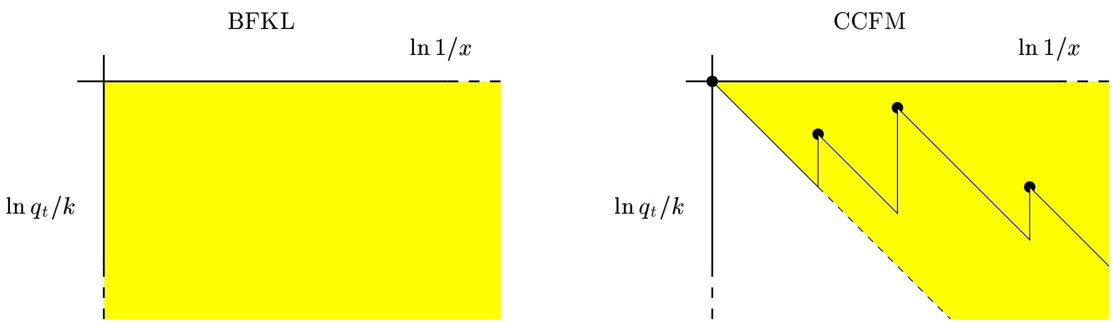

Figure 3: Phase-space available in the BFKL and CCFM cases. The shaded-area is the phase-space available to the first emission. In the CCFM case the black circles represent further emissions, and the lines the corresponding subsequent delimitation of the phase-space for future emissions.

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proportional to $\ln^2 x$, and given that one expects a number of emissions per unit area to be proportional to α_s, one sees immediately the well-known result [4] that there is a difference between the BFKL and CCFM predictions at the level of $\alpha_s \ln^2 x$.

If one introduces a resolvability cutoff, as done for example by Forshaw and Sabio Vera [5] (let μ_R be the resolution scale), then the difference in available phase-space for a single emission can never be larger than $\ln^2 k/\mu_R$: one loses the double logarithm in x, and the difference between the BFKL and CCFM results is now subleading, $\alpha_s \ln^2 k/\mu_R$.

This is the situation for a single CCFM emission. It is also possible to study the asymptotic properties of the final state.

The number of emissions with (rescaled) transverse momentum q_g and momentum fraction x_g contributing to the evolution of the gluon density to a point x, k is given by

$$\frac{dn(x,k,x_g,q_g)}{d\ln x_g\, d^2\vec{q}_g}\mathcal{A}(x,k,k) = \sum_{n=0}^{\infty}\prod_{i=1}^{n}\left(\bar{\alpha}_s\int\frac{dz_i}{z_i}\int\frac{d^2\vec{q}_i}{\pi q_i^2}\Delta(z_i,k_i,q_i)\Theta(q_i - z_{i-1}q_{i-1})\times \right.$$
$$\left. \times\,\Theta(q_{t,i}-\mu)\right)\frac{\bar{\alpha}_s}{\pi}\Theta\!\left(z_\Pi - \frac{x}{x_g}\right)\mathcal{A}(x_g,k_g,q_g)\Delta(z_0,k_0,q_g)\,. \tag{2.17}$$

where z_Π is defined in analogy with ζ_Π in (2.7) and

$$\vec{k}_i = \vec{k} + \sum_{j=i+1}^{n}\vec{q}_{t,j}\,, \qquad \vec{k}_g = \vec{k}_0 + \vec{q}_{t,g}, \qquad z_0 = \frac{x}{x_g z_\Pi}\,, \qquad q_0 \equiv q_g\,. \tag{2.18}$$

To simplify the notation, n has been taken for $p = k$. For $x \ll x_g$, the dependence on p would in any case be entirely contained in the $\mathcal{A}(x,k,p)$ factor.

In the case of nothing but k-conserving hard emissions (2.17) simplifies to

$$\frac{dn(x,k,x_g,q_g)}{d\ln x_g\, d\ln q_g}\mathcal{A}(x,k,k) = 2\bar{\alpha}_s\mathcal{A}(x_g,k,q_g)\sum_{n=0}^{\infty}\prod_{i=1}^{n}\left(\bar{\alpha}_s\int\frac{dz_i}{z_i}\int^{k}\frac{d^2\vec{q}_i}{\pi q_i^2}\Delta(z_i,k,q_i)\times\right.$$
$$\left.\times\;\Theta(q_i - z_{i-1}q_{i-1})\,\Theta(q_i-\mu)\right)\Theta(1-z_0)\,\Delta(z_0,k,q_g). \tag{2.19}$$

Making use of (2.14), and the fact that in the absence of k-changing emissions $\mathcal{A}(x,k,k)$ is independent of x, we "evaluate" all the integrals, and drop the explicit x and k dependence in n to obtain

$$\frac{dn(x_g,q_g/k)}{d\ln x_g\, d\ln q_g} = 2\bar{\alpha}_s\frac{\mathcal{A}(x_g,k,q_g)}{\mathcal{A}(x_g,k,k)}\,. \tag{2.20}$$

It has been shown in [2, 10] that asymptotically,

$$\frac{\mathcal{A}(x,k,q)}{\mathcal{A}(x,k,k)} \simeq e^{-\bar{\alpha}_s\ln^2 k/q}\,, \qquad q \ll k\,. \tag{2.21}$$

In the absence of k-changing emissions, this form is an exact eigenfunction of the DL evolution equation. Hence we arrive at the result that

$$\frac{dn(x_g, q_{t,g}/k)}{d\ln x_g \, d\ln q_{t,g}} = 2\bar{\alpha}_{\mathrm{s}} e^{-\bar{\alpha}_{\mathrm{s}} \ln^2 k/q_{t,g}}\,, \tag{2.22}$$

where use has been made of the equivalence between q_t and q at this accuracy. The first thing to note is that it differs from the BFKL result by subleading terms $\alpha_{\mathrm{s}}(\alpha_{\mathrm{s}} \ln^2 \frac{k}{q_{t,g}})^n$. By looking at the differential distribution with respect to $\ln q_{t,g}$, we have effectively introduced a resolvability cutoff, and as discussed earlier in the single emission case, this ensures the absence of double logarithms of x.

If on the other hand we integrate over all $q_{t,g}$, we have that

$$\frac{dn(x_g)}{d\ln x_g} = 2\bar{\alpha}_{\mathrm{s}} \int^k \frac{dq_{t,g}}{q_{t,g}} \frac{dn(x_g, q_{t,g}/k)}{d\ln x_g \, d\ln q_{t,g}} = \sqrt{\pi \bar{\alpha}_{\mathrm{s}}}\,. \tag{2.23}$$

The total number of emissions in a region $1 > x_g > x$ is then

$$n(x_g > x) \simeq \sqrt{\alpha_{\mathrm{s}} \ln^2 x}\,, \tag{2.24}$$

which is a double-logarithm in x.

So there is a close connection between double logarithms in x, and (formally subleading) double logarithms in q_t. The latter must be retained if one wants to take the limit of $\mu_R \to 0$. To put it in a different way, if one is interested in values of $q_{t,g}$ sufficiently low that that $\bar{\alpha}_{\mathrm{s}} \ln^2 k/q_{t,g} \gtrsim 1$, then subleading double logarithms in q must be resummed, and lead to a significant difference between the CCFM ($z \to 0$ divergent part only) and BFKL predictions.

3. CCFM with soft emissions

3.1 Factoring out soft emissions

So far we have examined the CCFM equation with only the $1/z$ part of the splitting function. For brevity, emissions produced by the $1/(1-z)$ part of the splitting function will be referred to as 'soft' emissions — they being soft relative to the exchanged gluon off which they are emitted.

The version of the CCFM equation with soft emissions has been examined relatively little, apart from its use for phenomenology in the SMALLX program [11], most recently studied in [12]. Part of the reason is the considerable technical difficulty involved, partly also uncertainty about exactly how the soft emissions are best implemented.

In the leading-logarithmic limit it turns out that these difficulties disappear, or become irrelevant, since they are mostly related to subleading issues.

The branching equation including the soft emissions is

$$\mathcal{A}(x,k,p) = \sum_{n=0}^{\infty}\prod_{i=1}^{n}\left(\bar{\alpha}_{\mathrm{s}}\int dz_i \frac{d^2\vec{q}_i}{\pi q_i^2}\left(\frac{1}{z_i}+\frac{1}{1-z_i}\right)\frac{\Delta_S(q_i)}{\Delta_S(z_{i-1}q_{i-1})}\,\Delta(z_i,k_i,q_i)\,\times\right.$$
$$\left.\times\,\Theta(q_i - z_{i-1}q_{i-1})\,\Theta(q_{t,i}-\mu)\right)\delta\left(x-\prod_{i=1}^{n} z_i\right)\times$$
$$\times\,\delta^2\!\left(\vec{k}+\sum_{i=1}^{n}\vec{q}_{t,i}\right)\frac{\Delta_S(p)}{\Delta_S(z_nq_n)}\Theta(p-z_nq_n)\,, \qquad (3.1)$$

where Δ_S, known as the Sudakov form factor, is given by

$$\ln\Delta_S(p) = -2\bar{\alpha}_{\mathrm{s}}\int\frac{dz}{1-z}\int\frac{dq}{q}\,\Theta(q_t-\mu)\,\Theta(p-q)\,, \qquad p>\mu\,, \qquad (3.2)$$

and $z_0q_0=\mu$. The k_i are defined as in (2.3).

In analogy with what was done above, one can take the double logarithmic limit of (3.1) to obtain

$$\mathcal{A}(x,k,p) = \sum_{n=0}^{\infty}\prod_{i=1}^{n}\left(\bar{\alpha}_{\mathrm{s}}\int dz_i \frac{d^2\vec{q}_i}{\pi q_i^2}\left(\frac{1}{z_i}+\frac{1}{1-z_i}\right)\frac{\Delta_S(q_i)\,\Delta(z_i,k_i,q_i)}{\Delta_S(z_{i-1}q_{i-1})}\,\times\right.$$
$$\left.\times\,\Theta(q_i - z_{i-1}q_{i-1})\Theta(q_{t,i}-\mu)\right)\delta\left(x-\prod_{i=1}^{n} z_i\right)\times$$
$$\times\,\delta^2\!\left(\vec{k}+\max(\vec{q}_{t,1},\ldots,\vec{q}_{t,n})\right)\frac{\Delta_S(p)}{\Delta_S(z_nq_n)}\Theta(p-z_nq_n)\,. \quad (3.3)$$

The k_i must be redefined as in (2.5).

We now examine how to separate the soft emissions from the others, much in the same way as was done for separating the k-conserving emissions from the k-changing emissions.

First, for all soft emissions we set $z_i=1$ everywhere except in the $1/(1-z_i)$ factor (remembering that $\Delta(1,k,q)=1$). The error that arises from this cannot be larger than next-to-leading, and will not be enhanced by double logarithms. We then observe that

$$1 = \sum_{m=0}^{\infty}\prod_{\ell=1}^{m}\left(\bar{\alpha}_{\mathrm{s}}\int\frac{d^2\vec{\rho}_\ell}{\pi\rho_\ell^2}\frac{d\zeta_\ell}{1-\zeta_\ell}\frac{\Delta_S(\rho_\ell)}{\Delta_S(\rho_{\ell-1})}\,\Theta(\rho_\ell-\rho_{\ell-1})\,\Theta(\rho_{t,\ell}-\mu)\right)\times$$
$$\times\,\frac{\Delta_S(P)}{\Delta_S(\rho_m)}\Theta(P-\rho_m)\,, \qquad (3.4)$$

for any value of $P>\rho_0$. When inserting this sum between two 'hard' emissions (say

q_a and q_b), it should be understood that ρ_0 means $z_a q_a$ and that P means q_b. To see that one can safely remove or insert such a sum between pairs of hard emissions (a, b) without changing the underlying structure (the 'backbone') of the chain one observes that all $\rho_{t,\ell} < q_{t,b}$. If b is a k-conserving emission, then so are all the soft emissions inserted before it. If b is a k-changing emission, then the soft emissions can also change k. Two things prevent this from becoming a problem. Firstly, the condition $\rho_{t,\ell} < q_{t,b}$ ensures that the rest of the chain is not affected by the insertion. Secondly, the soft emissions are not themselves affected by the value of k (the non-Sudakov form factor is always 1, since $z = 1$), so a change in k has no effect on them.

Therefore in the calculation of the cross section one can simply use (2.13). The final state is then determined by inserting sets of soft emissions (3.4) between every pair of hard emissions (and also before the first one, and after the last one).

3.2 Pattern of soft emissions

To understand the effect of soft emissions on the final state, let us consider just the contribution from the soft emissions between two hard emissions a and b, as in figure 4. From (3.4) the soft emissions are uniformally distributed, with mean density

$$\left\langle \frac{dn}{d\ln q_t \, d\ln 1/x} \right\rangle = 2\bar{\alpha}_{\mathrm{s}} \tag{3.5}$$

in the shaded region, whose shape is determined as follows: a soft emission ℓ has a momentum fraction $x_\ell = (1-\zeta_\ell)x_b \ll x_b$, leading to the vertical boundary; the horizontal lower boundary comes form the collinear cutoff: $\rho_{t,\ell} > \mu$; finally the two diagonal boundaries come from the angular ordering conditions:

$$q_{t,a}\frac{x_b}{x_a} < \rho_i = (1-z_i)\rho_{t,i} < q_{t,b}\,, \tag{3.6}$$

where, as before, factors of $1 - z_a$ and $1 - z_b$ have been approximated as 1, with the error being subleading.

One notes that in the region where the soft emissions are allowed, the density of emissions (3.5) is the same as the BFKL density (2.16).

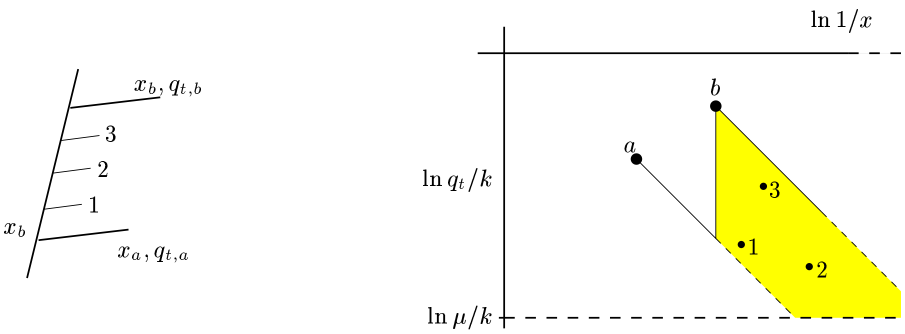

Figure 4: Illustration of the insertion of soft emissions ($\ell = 1, 2, 3$) between two hard emissions (a, b). The shaded area is the phase-space available to the soft emissions.

3.3 Combination of soft and hard emissions

Here we will see that the BFKL pattern of emissions is identical to that from the CCFM (hard + soft) equation. The fundamental point in determining this equivalence will be that the "order of the emissions" is not an observable property. It is only their final distribution in rapidity and transverse momentum that matters.

Let us consider the first part of a chain, containing two k-changing emissions, a and d (figure 5). These are identical in BFKL and CCFM, from the results of section 2. Emission a has $x = 1$.

In the BFKL case, the distribution for the next k-conserving emission that is ordered in angle with respect to the first one (call it b) is given by

$$2\bar{\alpha}_s \frac{dq_{t,b}}{q_{t,b}} \Delta(x_b, k_a, q_{t,a}) \Theta\left(q_{t,b} - x_b q_{t,a}\right). \tag{3.7}$$

The non-Sudakov form factor arises simply through a calculation of the probability of there not having been an angular ordered emission with momentum fraction larger than x_b. Equation (3.7) is identical to the distribution for the next hard (k-conserving) emission in the CCFM case.

Analogously, it is straightforward to determine that the distribution of the next k-conserving angular-ordered emission (in the BFKL case), c, is identical to that of the next hard emission (CCFM); and similarly for the probability of there being no angular-ordered (hard) k-conserving emissions in the BFKL (CCFM) case, before d.

So far we have only accounted for some of the emissions: in the BFKL case there are the emissions between a and b which are not angularly ordered with respect to a. They occupy the shaded region labelled A; the emissions between b and c that are not angularly ordered with respect to b occupy the region B, and so on. These emissions are independent and have the usual mean density (2.16).

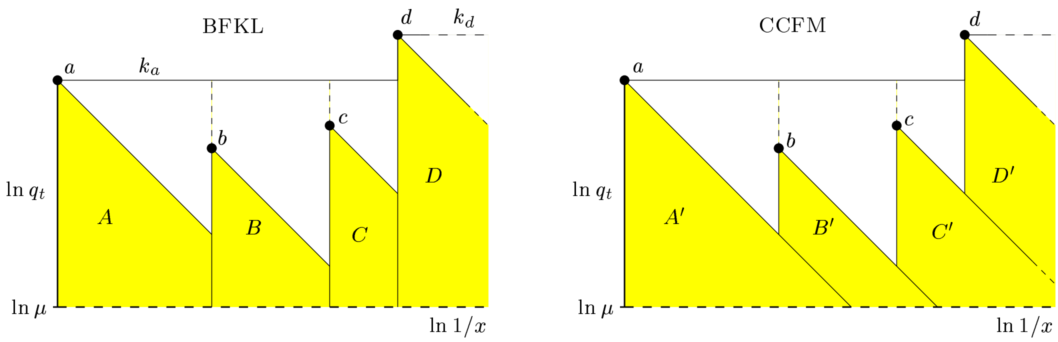

Figure 5: DL equivalence between BFKL and CCFM final states. In the BFKL case, the black discs are those emissions which form an angular-ordered set. The shaded regions contain the remaining, unordered, emissions, which are independent, with mean density $2\bar{\alpha}_s$. In the CCFM case, the black discs are hard emissions, while the shaded regions contain the soft emissions, which are independent, with mean density $2\bar{\alpha}_s$.

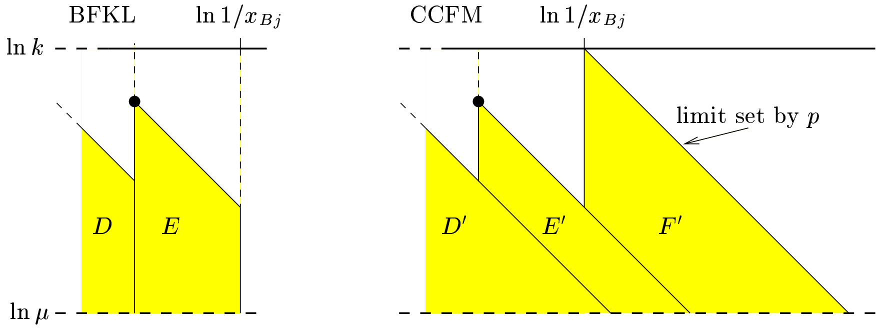

Figure 6: A comparison between the end of a BFKL and the end of a CCFM chain. The CCFM chain has the maximum-angle limit set by $p = k$. As before, the shaded region contains independent emissions with mean density $2\bar{\alpha}_s$.

In the CCFM case, there are the soft emissions. Those that come before a occupy the region labelled A' (they are independent, with mean density (3.5)). The soft emissions between a and b occupy region B', those between b and c, the region C', and so forth.

The sub-division of the "non-ordered regions", A–D in the BFKL case, and the soft regions A'–D' in the CCFM case is different. But the combination of A–D is identical to the that of A'–D'. Hence CCFM (hard + soft) gives the same pattern of emissions as BFKL.

This equivalence holds at the beginning and in the middle of the chain. There is one place where a difference between BFKL and CCFM double-logarithmic final states does arise, as illustrated in figure 6.

If BFKL evolution is carried out up to some limiting x, say x_{Bj}, there are no emissions with $x_g < x_{Bj}$. This corresponds to a vertical line cutting off the emissions at $x = x_{Bj}$ in figure 6. In the CCFM case, we will have the same vertical line limiting the hard emissions. But the soft emissions are delimited to the right only by diagonal lines (and by the intersection with the collinear cutoff μ). In the particular case shown, where the limiting angle is defined by $p = k$, the difference corresponds to a triangle in x, q_t space, containing on average $\bar{\alpha}_s \ln^2 p/\mu$ independent emissions.

This difference is formally subleading since it does not contain any $\ln x_{Bj}$ factors. In section 2.3 we saw that formally subleading DL corrections can get promoted to affect the leading DLs. This is not the case here though, since the effects of the difference are confined to one end of the evolution chain, and so get proportionately less important as one increases $\ln x_{Bj}$. For example the total multiplicity in a chain where k is determined by the first (hard) emission is $2\bar{\alpha}_s \ln 1/x_{Bj} \ln k/\mu$ in the BFKL case and $\bar{\alpha}_s(2 \ln 1/x_{Bj} \ln k/\mu + \ln^2 k/\mu)$ in the CCFM case ($p = k$). The difference is of relative order $\mathcal{O}\left(\ln k/\mu / \ln 1/x_{Bj}\right)$ and hence negligible.

Depending on the kind of initial condition, it is also possible for such differences to arise at the beginning of the chain. Given a suitably perverse initial condition they can even be of the order of a single $\alpha_s \ln^2 x$ term (by cutting out an initial triangle of area $\ln^2 x$) — however, again, they do not resum, and so do not give rise to a whole series of double logarithms.

4. Single logarithmic accuracy

In this section we refine the techniques used above, in order to demonstrate that BFKL and CCFM final states are equivalent at leading (single) logarithmic accuracy.

The source of error in the above sections comes from the definition of a k-conserving emission as any emission having $q_{t,i} < k_{i-1}$. In reality a k-conserving emission should have $q_{t,i} \ll k_{i-1}$. Thus there is a region of phase-space for each emission, of size $\mathcal{O}(\ln 1/x \ln 1/\epsilon)$, which is mistreated by the removal emissions which are not quite k-conserving. Here, the parameter ϵ has been introduced to define what is meant by $q_{t,i} \ll k_{i-1}$, namely $q_{t,i} < \epsilon k_{i-1}$. Taking into account the emission density proportional to $\bar{\alpha}_s$, one sees that for each emission one is mistreating a contribution of $\mathcal{O}(\bar{\alpha}_s \ln 1/x \ln 1/\epsilon)$, which is LL.

To extend the accuracy to be leading-logarithmic, one should therefore repeat the analysis of the above sections using a proper definition of a k-conserving emission. The basic procedure, as before, will be to divide the emissions into two sets: a 'backbone', consisting of those emissions that are not k-conserving, and which therefore may affect to the cross section. And those that are k-conserving, and which affect only the final state. The proof of the equivalence of BFKL and CCFM final states then relies firstly on the BFKL and CCFM ensembles of backbones being identical to LL accuracy; and secondly on the BFKL or CCFM rules for the addition of the k-conserving emissions to a given backbone having the same effect.

In the BFKL case, the backbone of k-changing emissions ($q_{t,i} > \epsilon k_{i-1}$) is given by

$$\mathcal{F}(x,k) = \sum_{n=0}^{\infty} \prod_{i=1}^{n} \left(\bar{\alpha}_s \int \frac{dz_i}{z_i} \int_\mu \frac{d^2\vec{q}_{t,i}}{\pi q_{t,i}^2} \Theta(q_{t,i} - \epsilon k_{i-1}) \Delta_{\epsilon k_i}(z_i, k_i) \right) \times \\ \times \delta\!\left(x - \prod_{i=1}^{n} z_i \right) \delta^2\!\left(\vec{k} + \sum_{i=1}^{n} \vec{q}_{t,i} \right), \tag{4.1}$$

where the cutoff has been introduced also in the form factor:

$$\ln \Delta_{\epsilon k}(z,k) = -2\bar{\alpha}_s \ln \frac{1}{z} \ln \frac{k}{\max(\mu, \epsilon k)}\,. \tag{4.2}$$

A sum over k-conserving emissions is then introduced after each backbone emission:

$$1 = \sum_{m=0}^{\infty} \prod_{\ell=1}^{m} \left(\bar{\alpha}_s \int \frac{d\zeta_\ell}{\zeta_\ell} \int_\mu^{\epsilon k_i} \frac{d^2\vec{\rho}_{t,\ell}}{\pi \rho_{t,\ell}^2} \Delta^{(\epsilon k_i)}(\zeta_\ell, k_i) \right) \Delta^{(\epsilon k_i)}(z_i/\zeta_\Pi, k_i)\, \Theta(\zeta_\Pi - z)\,, \tag{4.3}$$

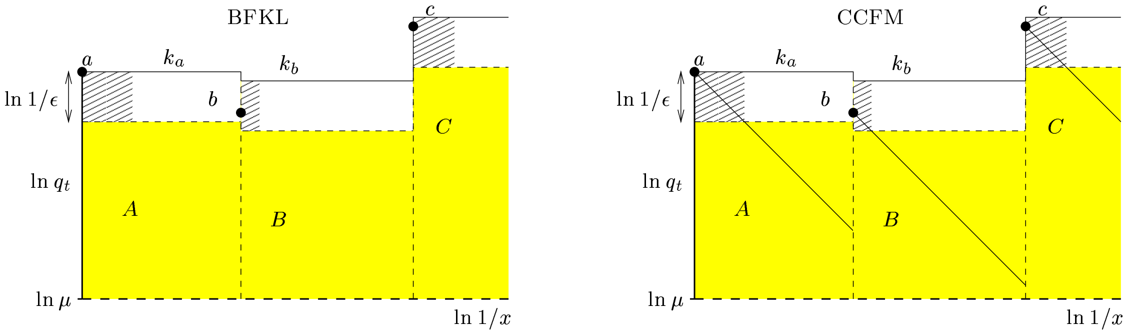

Figure 7: Breaking BFKL and CCFM evolution into a backbone plus k-conserving emissions — LL accuracy. Black dots are backbone emissions (the diagonal lines extending from them in the CCFM case indicate the angular ordering constraint for subsequent emissions). The shaded regions contain k-conserving emissions. The hashed regions indicate where the BFKL and CCFM emission densities may differ significantly (they should be understood to extend to $q_t \to \infty$).

with $\zeta_{\rm II}$ defined in analogy with (2.7) and

$$\Delta^{(\epsilon k)}(z,k) = \frac{\Delta(z,k)}{\Delta_{\epsilon k}(z,k)}\,. \tag{4.4}$$

It is safe to neglect the contribution of the ρ_ℓ's to the vector sum in (4.1) because their introduction can be compensated for by modifying each of the $q_{t,i}$ by a relative amount $\mathcal{O}(\epsilon)$. The k-conserving emissions fill up the shaded regions in figure 7, with the usual density of $2\bar\alpha_{\rm s}$ independent emissions per unit of rapidity and $\ln q_t$.

In the CCFM case, let us first consider a backbone with just hard emissions (even though some soft emissions might be more naturally classified as belonging to the backbone):

$$\begin{aligned}\mathcal{A}(x,k,p) = \sum_{n=0}^{\infty}\prod_{i=1}^{n}\Bigg(\bar\alpha_{\rm s}\int\frac{dz_i}{z_i}\int\frac{d^2\vec q_i}{\pi q_i^2}\Delta_{\epsilon k_i}(z_i,k_i,q_i)\Theta(q_i-\epsilon k_{i-1})\Theta(q_i - z_{i-1}q_{i-1})\,\times \\ \times\,\Theta(q_{t,i}-\mu)\Bigg)\Theta(p-z_nq_n)\,\delta\left(x-\prod_{i=1}^{n}z_i\right)\,\delta^2\!\left(\vec k+\sum_{i=1}^{n}\vec q_{t,i}\right)\end{aligned} \tag{4.5}$$

The modified non-Sudakov form factor is

$$\ln\Delta_{\epsilon k}(z,k,q) = -2\bar\alpha_{\rm s}\int_z^1\frac{d\zeta}{\zeta}\int\frac{d\rho}{\rho}\,\Theta(k-\rho)\,\Theta(\rho-\zeta q)\,\Theta(\rho-\mu)\,\Theta(\rho-\epsilon k)\,. \tag{4.6}$$

We then note that for $z < q/\epsilon k$ the BFKL and CCFM form factors differ only by a constant subleading factor,

$$\Delta_{\epsilon k}(z,k) = \Delta_{\epsilon k}(z,k,q)\,\exp\left(-\bar\alpha_{\rm s}\ln^2\frac{q_t}{\epsilon k}\right), \tag{4.7}$$

and that for $z_i > q_{t,i}/\epsilon k_i$ the phase-space limits for the $q_{t,i+1}$ integration become equal in the BFKL and CCFM cases, namely

$$\frac{d^2\vec{q}_{t,i+1}}{\pi q_{t,i+1}^2}\,\Theta(q_{t,i+1} - \epsilon k_i)\,. \tag{4.8}$$

So we are now in a position to show that the BFKL and CCFM equations lead to ensembles of backbones which are identical at LL accuracy: from both ensembles we remove backbones containing branchings with $z_i > q_{t,i}/\epsilon k_i$ (i.e. emissions falling into the hashed regions of figure 7). Given that typically[3] $q_{t,i} \sim k_i$, the likelihood of a branching violating this condition contains a factor $\bar{\alpha}_s \ln^2 \epsilon$. Hence a subleading fraction of backbones in each ensemble is removed. There is a one-to-one correspondence between the remaining backbones in the BFKL and CCFM ensemble. For a given backbone, the associated weights differ between BFKL and CCFM due to the differences between the form factors, (4.7), but again only by a subleading amount.

One might worry that since ϵ is a small parameter, $\bar{\alpha}_s \ln^2 \epsilon$ may not be a very 'respectable' subleading correction. The other source of inaccuracy is corrections of $\mathcal{O}(\epsilon)$. So it suffices to take $\epsilon = \bar{\alpha}_s$ for both quantities to be truly subleading.

That the backbones should be the same at LL level could have been guessed at right from the start, since the ensemble of backbones is responsible for the determining the cross section, and we know that the BFKL and CCFM cross sections differ only at subleading level.

The next stage in the study of the CCFM final state is the addition of the k-conserving emissions. The pattern for the hard k-conserving emissions inserted after every backbone emissions is much as in (2.14),

$$\begin{aligned} 1 = \sum_{m=0}^{\infty} \prod_{\ell=1}^{m} \left(\bar{\alpha}_s \int \frac{d\zeta_\ell}{\zeta_\ell} \int^{\epsilon k_i} \frac{d^2\vec{\rho}_\ell}{\pi \rho_\ell^2}\, \Delta^{(\epsilon k_i)}(\zeta_\ell, k_i, \rho_\ell)\, \Theta(\rho_\ell - \zeta_{\ell-1}\rho_{\ell-1})\, \Theta(\rho_\ell - \mu) \right) \times \\ \times\, \Delta^{(\epsilon k_i)}(z_i/\zeta_\Pi, k_i, \rho_0)\, \Theta(\zeta_\Pi - z_i)\,, \end{aligned} \tag{4.9}$$

with the appropriate modification of the upper limit and of the non-Sudakov form factor,

$$\Delta^{(\epsilon k)}(z, k, q) = \frac{\Delta(z, k, q)}{\Delta_{\epsilon k}(z, k, q)}\,. \tag{4.10}$$

The soft emissions are then to be inserted before each hard emission i, and after the

[3] In cases where $q_{t,i} \gg k_i$ there is an additional subleading, but double-logarithmic price to pay, $\sim \bar{\alpha}_s \ln^2 q_{t,i}/k_i$. In cases where $q_{t,i+1} \gg k_i$ the removed region will correspond to a correction $\sim \bar{\alpha}_s \ln \epsilon \ln q_{t,i+1}/k_i$, which is also subleading.

last one (in which case $q_i \equiv p$)

$$1 - \mathcal{O}\left(\bar{\alpha}_{\rm s} \ln^2 \epsilon\right) = \sum_{m=0}^{\infty} \prod_{\ell=1}^{m} \left(\bar{\alpha}_{\rm s} \int \frac{d^2\vec{\rho}_\ell}{\pi \rho_\ell^2} \frac{d\zeta_\ell}{1-\zeta_\ell} \frac{\Delta_S(\rho_\ell)}{\Delta_S(\rho_{\ell-1})} \Theta(\rho_\ell - \rho_{\ell-1}) \times \right.$$
$$\left. \times\, \Theta(\epsilon k_j - \rho_\ell)\, \Theta(\rho_{t,\ell} - \mu) \right) \frac{\Delta_S(p)}{\Delta_S(\rho_n)} \Theta(q_i - \rho_m)\,, \quad (4.11)$$

where $\rho_0 \equiv z_{i-1} q_{i-1}$. For a soft gluon with momentum fraction x_s, the index j (of k_j) is given by the condition $x_{j-1} > x_s > x_j$. This leads to a slightly different condition from simple k-conservation: it ensures that the region in which soft emissions can be present is at most the shaded region of figure 7. This together with the limits on the z's in the backbone ensures that a soft emission does not affect the k after the next hard emission by more than a relative amount ϵ, and that the soft emissions do not occupy the hashed regions in figure 7. There is the price of a subleading contribution $\mathcal{O}\left(\bar{\alpha}_{\rm s} \ln^2 \epsilon\right)$ (corresponding to the area of the lower triangles in the hashed region in the CCFM diagram of figure 7) which arises from the incomplete cancellation between the real and virtual parts of (4.11), and which should be included in the weight of the backbone. At LL accuracy this is of no relevance.

Finally one needs to show that the combination of the k-conserving hard and soft emissions fills up the shaded region in figure 7 with a mean density of $2\bar{\alpha}_{\rm s}$ independent emissions per unit rapidity and $\ln q_t$, as in the BFKL case. The procedure for doing so is identical to that used in section 3.3, but with appropriate modifications of the limits on the emitted transverse momenta. There is therefore no need to reproduce the explicit proof here. This completes the demonstration that BFKL and CCFM final states are identical at leading (single) logarithmic level.

5. Conclusions

The recurrent theme in this article has been that to study final-state properties it is useful to split emissions into those which change the exchanged transverse momentum ('backbone' emissions), and those which do not. The former, being responsible for determining the cross section, are almost bound to have the same pattern, since the BFKL and CCFM cross sections are identical at LL order.

It is in the treatment of the latter, the collinear emissions, that the BFKL and CCFM approaches at first sight appear as if they will lead to different results. In the CCFM case there are two types of collinear emissions, 'hard' ($z \to 0$) and 'soft' ($z \to 1$) ones. Only after their combination does one obtain collinear emissions with the same pattern as in BFKL. Differences seen in the literature between BFKL and CCFM final states were due to the inclusion of only the hard collinear emissions (i.e. soft CCFM emissions cancel the double logarithms of x from hard CCFM emissions).

It should be emphasised that it is only their leading-logarithmic predictions and not the BFKL and CCFM equations themselves (in the sense of their physical content) that are equivalent: the BFKL equation is derived in the limit of strong ordering in x (without coherence or soft emissions) — this is then somewhat arbitrarily extended to exact ordering. The CCFM derivation deals explicitly with the issues of coherence and soft radiation, as is necessary in order to guarantee the leading-logarithms of the final state.

A consequence of these differences is, for example, that in the BFKL equation, it is impossible to consider z in the usual DGLAP sense, since its value is largely determined, through the form factor, by the value of the collinear cutoff — for small cutoffs, z is close 1. The structure of the CCFM equation is much more amenable to a direct physical interpretation (z does have the usual DGLAP interpretation), and consequently perhaps a better starting point for the correct inclusion of subleading effects [13, 14] such as the full splitting function. As to whether or not this is the case will depend on whether other next-to-leading contributions can be correctly included and resummed (for discussions of important physical issues that need to be dealt with, the reader is referred to [15]).

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